Statistical Fuzzy Black-Scholes numbers as index of performance

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Abstract. In this paper, we use the set of propositions in our previous papers and the statistical data of the Italian Stock Exchange for having a normal and convex fuzzy statistical version of the Black-Scholes value where the risk free instantaneous interest intensity, the volatility and the initial stock price are statistical normal and convex fuzzy numbers. With our statistical Black-Scholes fuzzy numbers we define index of performance varying in the time which says that the market is becoming with more risks: the risk of loss increase and the forecast of profits decrease.

With respect to the probabilistic our fuzzy method is more simple and immediate for having forecasts on the financial market.

Keywords. Finance; Statistics of financial markets; European option; Black-Scholes formula; Fuzzy numbers.

1. Introduction

In this paper, we use the fuzzy theory of Zadeh, the set of propositions used in our previous papers and statistical data of Italian Stock Exchange to have a fuzzy version of the Black-Scholes (B-S) value for an European call option where the instantaneous intensity of the risk free interest, $\delta$, the standard deviation of the instantaneous intensity of return from the underlying security (volatility), $\sigma$, and the initial stock price, $a$, are fuzzy numbers.

With a theorem we deduce that our B-S fuzzy numbers are convex and normal.

Then we give a statistical example: with the data of the Italian MIB 30, i.e., its time series, we build a statistical triangular B-S fuzzy number which for every belief degree gives a B-S value. This fuzzy number varies in the time.

With the support of our statistical B-S fuzzy numbers we define two index of the performance of the market, one for loss and another for profits, which vary in the time. In particular, if we gamble on the increase of the prices of the components in Mib30, we see that in the future with a high belief degree the loss increase and the profit decrease.

The paper is organized as follows: in section 2 we set the preliminaries and give the theorems which links the Zadeh’s extension principle to the construction of convex numbers with cut-functions in accordance with the theorems in [1,2], [6] and Zadeh’s identity; in 3 we define three fuzzy numbers which we deduce by probabilistic densities to use for our statistical applications; in 4 we give our model of the Black-Scholes fuzzy convex numbers and its theorem and in 5 we put statistical applications and cues for further research.
2. Preliminaries and Theorems

Let \((\Omega, A, P)\) be a probability space, where \(\Omega\) is the set of real numbers, \(A = B(\Omega)\) is the Borel’s \(\sigma\)-field and \(P\) a probability measure.

We may identify a subset \(A\) of \(\Omega\) (event) with its characteristic function,

\[ \chi_A : \omega \in \Omega \rightarrow \chi_A(\omega) \in \{0, 1\}, \]

i.e., \(A\) is the support of \(\chi_A\):

\[ \chi_A(\omega) = \begin{cases} 1, & \omega \in A; \\ 0, & \omega \notin A. \end{cases} \]

The membership function, \(f\), is a generalization of \(\chi_A\) (see Zadeh [8])

\[ f : \omega \in \Omega \rightarrow f(\omega) \in [0, 1], \]

that is, \(0 \leq f(\omega) \leq 1\): the support of \(f\), i.e. \(\hat{A} = \{\omega \in \Omega : f(\omega) \neq 0\}\) is called fuzzy set or fuzzy number, f.n. \((\Omega = R)\); \(f(\omega)\) represents: i) a membership degree of \(\omega\) as element of \(\hat{A}\); ii) a belief degree that \(\omega\) happens.

\(f\) is called fuzzy number too.

The \(y\)-cut of \(f\), stated at the level \(y \in [0, 1]\), is defined by:

\[ C^f_y = \{\omega \in \Omega : f(\omega) \geq y\}. \]

We say that \(f\) is convex if all its \(y\)-cuts are closed and bounded real intervals.

\(f\) is normal if there exists \(\omega_0 \in \Omega\) such that \(f(\omega_0) = 1\).

We say that \(f \in \mathcal{N}\) if \(f\) is convex and normal.

If \(f \in \mathcal{N}\) then we have

\[ \forall y \in [0, 1], \quad C^f_y = [f^{-l}(y), f^{-r}(y)] \]

where \(f^{-l}(y)\) is increasing, \(f^{-r}(y)\) is decreasing, and by Zadeh’s identity we have:

\[ f(\omega) = \sup_{y \in [0, 1]} y \cdot \chi_{[f^{-l}(y), f^{-r}(y)]}(\omega) \quad (1) \]

where \(\chi\) is characteristic function of the real interval \([f^{-l}(y), f^{-r}(y)]\). We call \(f^{-l}(y)\) the left cut-function of \(f\) and \(f^{-r}(y)\) the right cut-function of \(f\). Besides, if the cut-functions are continuous, then \(f\) given by (1) is in \(\mathcal{N}\).

Zadeh’s extension principle: Let \(x := (x_1, \ldots, x_n) \in R^n\), \(h(x)\) a continuous nonfuzzy function with values in \(\Omega\):

\[ h : x \in R^n \rightarrow h(x) = \omega \in \Omega, \]

and \(n\) fuzzy numbers

\[ f_1, f_2, \ldots, f_n. \]

The Zadeh’s extension principle (Z.e.pr.) gives a belief degree to every value \(\omega = h(x)\), i.e., the Z.e.pr. defines a belief degree or membership function, \(\tilde{h}\), by

\[ \tilde{h}(\omega) := \sup_{x \in R^n : h(x) = \omega} \min\{f_1(x_1), \ldots, f_n(x_n)\}. \]

So, from the nonfuzzy function \(h\), we have that the Z.e.pr. induces a fuzzy valued function, i.e.,

\[ \tilde{h} : \omega = h(x_1, \ldots, x_n) \Rightarrow \tilde{h}(\omega) \in [0, 1] \]
and \( \tilde{h} \) is obtained by \( h \) and \( f_1, \ldots, f_n \), therefore we can denote \( \tilde{h} \) (defined by Z.e.pr.) in the following way:

\[
\tilde{h} = \tilde{h}(f_1, f_2, \ldots, f_n)
\]

**Proposition 1.** If \( f_1, \ldots, f_n \) are in \( \mathcal{N} \) and \( h \) is continuous nonfuzzy function, then (\( \forall y \in [0,1] \)) the \( y \)-cut of the fuzzy number \( \tilde{h}(f_1, \ldots, f_n) \) is equal to the \( h \) of the cuts of \( f_1, \ldots, f_n \):

\[
C_y^{\tilde{h}(f_1, \ldots, f_n)} = h(C_y^{f_1}, \ldots, C_y^{f_n}) = \{ \omega = h(x_1, \ldots, x_n) \in \Omega : x_1 \in C_y^{f_1}, \ldots, x_n \in C_y^{f_n} \}.
\]

This Proposition is an extension obtained by induction by a Proposition of Biacino-Lettieri (see 2.4 in [1]) and it is used in our previous papers [2, 6].

From Proposition 1 we have the following

**Proposition 2.** Given a continuous nonfuzzy function \( h(x_1, \ldots, x_n) \) increasing with respect to \( x_i \) and decreasing with respect to \( x_j \) and given the fuzzy numbers \( f_1, \ldots, f_n \) in \( \mathcal{N} \). Then we have that the membership function \( \tilde{h} \) is in \( \mathcal{N} \) if:

i) the left cut-function \( \tilde{h}^{-i}(y), y \in [0,1] \) is given by

\[
h^{-i}(y) = h(\ldots, f_i^{-i}(y), \ldots, f_j^{-r}(y), \ldots)
\]

and is increasing;

ii) the right cut-function \( \tilde{h}^{-r}(y), y \in [0,1] \) is given by

\[
h^{-r}(y) = h(\ldots, f_i^{-r}(y), \ldots, f_j^{-l}(y), \ldots)
\]

and is decreasing;

iii) \( h^{-i}(y) \leq h^{-r}(y) \) and the equal holds if \( y = 1 \),

then we have

\[
\tilde{h}(\omega) = \sup_{y \in [0,1]} y \cdot X_{[\tilde{h}^{-i}(y), \tilde{h}^{-r}(y)]}(\omega)
\]

is an element of \( \mathcal{N} \), where \( \chi \) is the characteristic function of the real interval \( [\tilde{h}^{-i}(y), \tilde{h}^{-r}(y)] \).

**Proof.** If \( f_1, \ldots, f_n \in \mathcal{N} \) then we have that \( \forall y \in [0,1] \) their cuts are the following compact (closed and limited) intervals, \( i \in \{1, \ldots n\} \):

\[
C_y^{f_i} = [f_i^{-l}(y), f_i^{-r}(y)].
\]

On the other hand, by Proposition 1, from the hypothesis of continuity of \( h \) it follows that \( C_y^{\tilde{h}(f_1, \ldots, f_n)} \) is a compact interval. So, if the left-cut function \( \tilde{h}^{-i}(y) \) is increasing and the right \( \tilde{h}^{-r}(y) \) is decreasing, \( \tilde{h}^{-i}(y) \leq \tilde{h}^{-r}(y) \) and \( \tilde{h}^{-i}(1) = \tilde{h}^{-r}(1) \), then \( C_y^{\tilde{h}} \) is equal to the following

\[
C_y^{\tilde{h}} = [\tilde{h}^{-i}(y), \tilde{h}^{-r}(y)]
\]

that is \( \tilde{h} \) is a normal and convex fuzzy number, i.e. \( \tilde{h} \) is in \( \mathcal{N} \) and by Zadeh’s identity we have that \( \tilde{h} \) has the following shape:

\[
\tilde{h}(\omega) = \sup_{y \in [0,1]} y \cdot X_{[\tilde{h}^{-i}(y), \tilde{h}^{-r}(y)]}(\omega)
\]
3. Examples of convex fuzzy numbers useful for our applications

For our fuzzy option price numbers we can use, for example, one of the following fuzzy numbers:

1) Triangular fuzzy numbers

\[ f(\omega) = \begin{cases} \frac{\omega - a}{b - a}, & \text{if } \omega \in [a, b]; \\ \frac{c - \omega}{c - b}, & \text{if } \omega \in [b, c]; \\ 0, & \text{elsewhere}. \end{cases} \]

where \( a < b < c \). The graph of \( f \) looks like a triangle. In our cases, the support \([a, c]\) may be calculated by a statistical time series and \( b \) may be equal to its arithmetical mean or the value of the time series with the bigger frequency.

In this case the y-cuts are given by:

\[ \forall y \in ]0, 1[, \quad [f^{-l}(y), f^{-r}(y)] \]

with

\[
\begin{align*}
  f^{-l}(y) &= a + (b - a)y; \\
  f^{-r}(y) &= c - (c - b)y.
\end{align*}
\]

2) Gaussian fuzzy numbers

\[ g(\omega) = e^{-\frac{(\omega - \mu)^2}{2\sigma^2}} \]

where the mean \( \mu \) and the standard deviation \( \sigma \) are calculated using a statistical time series.

\[ \forall y \in ]0, 1[ \text{ the cut-functions are given by:} \]

\[
\begin{align*}
  g^{-l}(y) &= \mu - \sqrt{-2\sigma^2 \log_e(y)}; \\
  g^{-r}(y) &= \mu + \sqrt{-2\sigma^2 \log_e(y)}.
\end{align*}
\]

The belief degree of \( \omega = \mu \) is equal to \( g(\mu) = 1 \) which is the max of \( g \). In fact the derivative of \( g \) is given by

\[ g'(\omega) = \left(\frac{-(\omega - \mu)}{\sigma^2}\right)e^{\frac{-1(\omega - \mu)^2}{2\sigma^2}} \]

which is equal to zero if \( \omega = \mu \). \( g \) has symmetric and campanulate graph.

The flex points are: \( \mu - \sigma, \mu + \sigma \). In fact the second derivative of \( g \) is given by

\[ g''(\omega) = -\frac{1}{\sigma^2}e^{\frac{-1(\omega - \mu)^2}{2\sigma^2}} - \frac{\omega - \mu}{\sigma^2}\left(\frac{1(\omega - \mu)}{\sigma^2}\right)e^{\frac{-1(\omega - \mu)^2}{2\sigma^2}} = -\frac{1}{\sigma^2}e^{\frac{-1(\omega - \mu)^2}{2\sigma^2}}[1 - \frac{(\omega - \mu)^2}{\sigma^2}] \]

\[ g''(\omega) = 0 \text{ if and only if } (\frac{1}{\sigma^2})(\omega - \mu)^2 = 1, \text{ i.e. } (\omega - \mu)^2 = \sigma^2 \leftrightarrow \omega - \mu = \pm \sigma \leftrightarrow \omega = \mu \pm \sigma. \]

3) Gaussian fuzzy number with compact support

\[ g(\omega) = \begin{cases} e^{-\frac{(\omega - \mu)^2}{2\sigma^2}}, & \text{if } \omega \in [\mu - 3.9\sigma, \mu + 3.9\sigma]; \\ 0, & \text{elsewhere} \end{cases} \]

where \( \mu \) and \( \sigma \) is calculated with the statistical time series and making arithmetical mean.
4. A Fuzzy Black-Scholes

Let \( k \) be the strike price, \( \delta \) the annual instantaneous intensity of the risk free interest, \( a \) the initial market price of the underlying security, \( t \) the expiration date, \( \sigma^2 \) the variance of the annual instantaneous intensity of the random return from the underlying security, then the classical Black-Scholes value for an European call option at time zero is given by

\[
c(a, \delta, \sigma) = aN(d_1) - ke^{-\delta t}N(d_2)
\]

where \( N(x) \) is the cumulative distribution function for a standard normal r.v.

\[
N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy,
\]

also

\[
d_1 = \frac{\log(\frac{a}{k}) + (\delta + \frac{1}{2}\sigma^2)t}{\sigma \sqrt{t}}
\]

and \( d_2 = d_1 - \sigma \sqrt{t} \).

Note that if the volatility \( \sigma \) is very big then we may have \( d_1 > 3.9 \) hence \( d_2 < -3.9 \). In this case the values of the cumulative distribution function of the standard normal r.v. become \( N(d_1) = 1, N(d_2) = 0 \) respectively and so we have that the price of the call is equal to the price of the underlying security at time zero (this is the extreme upper value of the call) (see [4, 5]).

The price of the European call (1) depends in particular on the initial market price of the underlying security, \( a \), and on the strike price, \( k \), which vary in the time. If we consider (1) varying in the time, then its fuzzy version gives us an index of performance as we see in the Proposition 3 and in 5 where we make statistical application on Mib30.

**Proposition 3. (on normal and convex fuzzy Black-Scholes numbers.)**

If we take these fuzzy numbers

\[
f_1 = \tilde{a}, f_2 = \tilde{\delta}, f_3 = \tilde{\sigma} \in \mathcal{N}
\]

instead of the corresponding crisp, then the fuzzy version of the Black-Scholes formula, \( \tilde{h} = \tilde{c} \), is in \( \mathcal{N} \) if and only if

i) the left cut-function of \( \tilde{c} \) given by

\[
\tilde{c}^{-l}(y) = a^{-l}(y)N(d_1^{-l}(y)) - ke^{-\delta t - l(y)}N(d_2^{-l}(y))
\]

is increasing;

ii) the right cut-function of \( \tilde{c} \) given by

\[
\tilde{c}^{-r}(y) = a^{-r}(y)N(d_1^{-r}(y)) - ke^{-\delta t - r(y)}N(d_2^{-r}(y))
\]

is decreasing;

iii) the left cut-function of \( \tilde{c} \) is less than the right:

\[
\tilde{c}^{-l}(y) \leq \tilde{c}^{-r}(y);
\]

Moreover, with the left and right cut-functions we may write \( \tilde{c} \):

\[
\tilde{c}(\omega) = \sup_{y \in [0,1]} y \cdot \chi_{\{c^{-l}(y) < \tilde{c}^{-r}(y)\}}(\omega),
\]

and \( \tilde{c} \) is an element of \( \mathcal{N} \).
Proof.

We suppose to know these three convex and normal fuzzy numbers
\[ f_1 = \tilde{a}, \ f_2 = \tilde{\delta}, \ f_3 = \tilde{\sigma} \in \mathcal{N} \]
that is, \( \forall x = (x_1, x_2, x_3) \in \mathbb{R}^3 \), we suppose to know the belief degree \( 0 \leq \tilde{a}(x_1) \leq 1 \) of the random event \( \{ a = x_1 \} \) and analogously we know the belief degree \( 0 \leq \tilde{\delta}(x_2) \leq 1 \) and \( 0 \leq \tilde{\sigma}(x_3) \leq 1 \) of the random events \( \{ \delta = x_2 \} \) and \( \{ \sigma = x_3 \} \) respectively.

Then we obtain the fuzzy price of the call, \( \tilde{c} \) by the Zadeh’s extension principle:
\[
\tilde{c}(\omega) := \sup_{x \in \mathbb{R}^3: x(x) = \omega} \min\{ \tilde{a}(x_1), \tilde{\delta}(x_2), \tilde{\sigma}(x_3) \}
\]
according to the Black-Scholes formula
\[
c(a, \delta, \sigma) = aN(d_1) - ke^{-\delta t} N(d_2).
\]
On the other hand, if we know the fuzzy numbers \( \tilde{a}, \tilde{\delta}, \tilde{\sigma} \in \mathcal{N} \) then we can write their left and right cut-functions
\[
\forall y \in [0, 1], \ \tilde{a}^{-l}(y), \tilde{\delta}^{-l}(y), \tilde{\sigma}^{-l}(y); \ \tilde{a}^{-r}(y), \tilde{\delta}^{-r}(y), \tilde{\sigma}^{-r}(y)
\]
and for Proposition 2 we have that for every fixed belief degree, \( y \in [0, 1] \), the cut functions of the fuzzy numbers \( \tilde{d}_1 \) and \( \tilde{d}_2 \) are given by:
\[
\tilde{d}_1^{-l}(y) = \log(\frac{\tilde{a}^{-l}(y)}{\tilde{\delta}^{-l}(y) + \frac{1}{2}(\tilde{\sigma}^{-l}(y))^2}) + \frac{2}{\sqrt{\tilde{a}^{-l}(y)}} \tilde{\sigma}^{-l}(y) \tilde{d}_2^{-l}(y) = \tilde{d}_1^{-l}(y) - \tilde{\sigma}^{-l}(y) \sqrt{t};
\]
\[
\tilde{d}_2^{-r}(y) = \frac{\tilde{a}^{-r}(y) + \frac{1}{2}(\tilde{\sigma}^{-r}(y))^2}{\tilde{\sigma}^{-r}(y)} \frac{1}{\sqrt{\tilde{a}^{-r}(y)}} \tilde{d}_2^{-r}(y) = \frac{\tilde{a}^{-r}(y) + \frac{1}{2}(\tilde{\sigma}^{-r}(y))^2}{\tilde{\sigma}^{-r}(y)} \frac{1}{\sqrt{\tilde{a}^{-r}(y)}} + \tilde{\sigma}^{-r}(y) \sqrt{t}.
\]
We see that the left cut-function is increasing because it is a sum of increasing functions and analogously the right cut-function is decreasing.

Besides, with all these cut-functions we build the left and right cut-functions of \( \tilde{c} \) given by:
\[
\tilde{c}^{-l}(y) = \tilde{a}^{-l}(y) N(\tilde{d}_1^{-l}(y)) - ke^{-\tilde{\delta}^{-l}(y) N(\tilde{d}_2^{-l}(y))}
\]
\[
\tilde{c}^{-r}(y) = \tilde{a}^{-r}(y) N(\tilde{d}_1^{-r}(y)) - ke^{-\tilde{\delta}^{-r}(y) N(\tilde{d}_2^{-r}(y))}
\]
Moreover we have \( \tilde{c}^{-l}(y) \leq \tilde{c}^{-r}(y) \) and by Zadeh’s identity
\[
\tilde{c}(\omega) = \sup_{y \in [0, 1]} y \cdot \chi_{[\tilde{c}^{-l}(y), \tilde{c}^{-r}(y)]}(\omega).
\]
For \( y = 0 \), we have that the support of the fuzzy Black-Scholes number \( \tilde{c} \) is
\[
\Omega = [\tilde{c}^{-l}(0), \tilde{c}^{-r}(0)].
\]
In particular we have that the real number \( \tilde{c}^{-l}(0) \) is negative if \( \tilde{a}^{-l}(0) << k \) and \( \tilde{c}^{-r}(0) \) is positive if \( \tilde{a}^{-r}(0) >> k \). The two prices
\[
\tilde{c}^{-l}(0), \tilde{c}^{-r}(0)
\]
represent the pessimistic and optimistic performance on the future if we gamble on the increase of the price.

The real numbers \( \tilde{c}^{-l}(0), \tilde{c}^{-r}(0) \) are two synthesis, they summarize the expectations in all the fuzzy parameters which we use, so they may represent index of the performance of the market of the underlying security. Obviously if this security is a market index, as the Mib30, then they may represent index of the performance of the market. Obviously these index varying in the time. We will see that its variations help us to evaluate and to forecast the future of the market.
5. Statistical convex fuzzy numbers

Here we build a fuzzy set using statistical observations of financial prices. We have a time series of the daily prices,

\[ p(t) \]
of the “MIB 30” (for 55 months) and we want to build a fuzzy option price for an option with strike price 39500, maturity on 11 days.

From the security time series of the price we build the time series of the instantaneous intensity of returns from the security (using continuous compounding),

\[ r(t) = \log \frac{p(t)}{p(t-1)} \]

Every month we have a daily arithmetical mean of \( r(t) \). So, from the observation of 55 months we have 55 means:

\[ \forall k \in \{1, 2, ..., 55\}, m(k) = \frac{1}{30} \sum_{t=1}^{30} r(t) \]

and 55 daily arithmetical variances of \( r_t \):

\[ \forall k \in \{1, 2, ..., 55\}, s^2(k) = \frac{1}{29} \sum_{t=1}^{30} (r(t) - m(k))^2 \]

So we have a time series of 55 standard deviations, i.e. the *volatility time series*:

\[ \forall k \in \{1, 2, ..., 55\}, s(k) = \sqrt{s^2(k)} \]

From this time series we may build a convex fuzzy number, \( \tilde{\sigma} \), to represent the fuzzy volatility. We can do this in many ways. One of these is to calculate the arithmetical mean and the variance of the volatility time series:

\[ \mu = \frac{1}{55} \sum_{k=1}^{55} s(k), \]

\[ \sigma = \frac{1}{54} \sum_{k=1}^{55} (s(k) - m(s))^2 \]

and to take the Gaussian fuzzy number with mean \( \mu \) and variance \( \sigma^2 \).

Another way is using a triangular fuzzy number to represent the fuzzy volatility. In this case we need to know the support. We can do this reading the time series. In the case of the “MIB 30” volatility we have:

1) the min daily variance of the volatility in August 2000: 0.000068;
2) the max daily variance of the volatility in September 2002: 0.000521.

We have to multiply by 365 for obtaining the min variance per annum:

\[ 0.000068 \times 365 = 0.02482 \]

and the max variance per annum:

\[ 0.000521 \times 365 = 0.190165 \]

The min and max standard deviations per annum were

\[ \sqrt{0.02482} = 0.1575, \quad \sqrt{0.190165} = 0.436079 \]

respectively.
So we may take the interval \([0.1575, 0.436079]\) as the support of a triangular fuzzy annual volatility of the security "MIB 30":

\[
\tilde{\sigma}(\omega) = \begin{cases} 
\frac{0.15 - \omega}{0.10}, & \text{if } \omega \in [0.15, 0.25] \\
\frac{0.50 - \omega}{0.25}, & \text{if } \omega \in [0.25, 0.50] \\
0, & \text{elsewhere.}
\end{cases}
\]

The y-cuts are given by:

\[
\forall y \in [0, 1], \quad [\tilde{\sigma}^{-l}(y), \tilde{\sigma}^{-r}(y)]
\]

\[
\tilde{\sigma}^{-l}(y) = 0.15 + 0.10y; \\
\tilde{\sigma}^{-r}(y) = 0.50 - 0.25y.
\]

So with belief degree \(y = 0.90\) we have that the annual volatility is in the real interval \(C_{y}^\tilde{\sigma} = [0.24, 0.725]\) because:

\[
\tilde{\sigma}^{-l}(0.90) = 0.15 + 0.10(0.90) = 0.24; \\
\tilde{\sigma}^{-r}(0.90) = 0.50 - 0.25(0.90) = 0.275.
\]

For building the fuzzy instantaneous intensity of the risk-free interest rate for the same duration of the option, we read:

1) the risk-free interest rates of the Italian Banks in the real interval \(i_1 \in [0.015, 0.018]\) then it follows that \(\delta_1 \in [0.0149, 0.017]\);

2) the interest rates of the "Buoni Ordinari del Tesoro" \(i_2 \in [0.02, 0.025]\) \(\Rightarrow \delta_2 \in [0.0198, 0.02469]\).

So, we give a great belief degree to these data if we take the basic points of the triangle in this way:

\[b = \tilde{\delta}^{-l}(1) = 0.02, \quad a = 0.014 = \tilde{\delta}^{-l}(0), c = 0.025 = \tilde{\delta}^{-r}(0)\]

so the support of the triangular fuzzy number \(\tilde{\delta}\) is the interval \([0.014, 0.025]\):

\[
\tilde{\delta}(\omega) = \begin{cases} 
\omega - 0.014, & \text{if } \omega \in [0.014, 0.02] \\
0.025 - \omega, & \text{if } \omega \in [0.02, 0.025] \\
0, & \text{elsewhere.}
\end{cases}
\]

The y-cut functions are:

\[
\tilde{\delta}^{-l}(y) = 0.014 + 0.006y; \\
\tilde{\delta}^{-r}(y) = 0.025 - 0.005y.
\]

For building the today fuzzy price (October 30th, 2006) of the "MIB 30" we see in the Milano Stock Exchange the prices today are max 40000 and min 39500. We might use these prices to build the support of the fuzzy price \(\tilde{a}\) and to build the f.n. as triangular. But, we want put in this numbers also our evaluations on the future so we take the min equal 34000 and the max 43000. So, we give a belief degree to these evaluations and observations defining a triangular \(\tilde{a}\) in this way:

\[
\tilde{a}(\omega) = \begin{cases} 
\frac{34000 - \omega}{6000}, & \text{if } \omega \in [34000, 40000] \\
\frac{43000 - \omega}{3000}, & \text{if } \omega \in [40000, 43000] \\
0, & \text{elsewhere.}
\end{cases}
\]

The y-cut functions are:

\[
\tilde{a}^{-l}(y) = 34000 + 6000y; \\
\tilde{a}^{-r}(y) = 43000 - 3000y.
\]

With belief degree \(y = 1\) we have that today the Mib30 is equal to \(\tilde{a}^{-l}(1) = \tilde{a}^{-r}(1) = 40000\). The values 34000 and 43000 have membership degree equal to zero. But if we want to make evaluations on the future we may say that in the future with belief degree \(y = 0.5\) the min and max price of Mib30 will be equal to:

\[
\tilde{a}^{-l}(0.50) = 34000 + 6000 \cdot 0.50 = 37000; \\
\tilde{a}^{-r}(0.50) = 43000 - 3000 \cdot 0.50 = 41500.
\]
Statistical fuzzy Black-Scholes

Using the triangular fuzzy numbers $\tilde{a}, \tilde{\delta}, \tilde{\sigma}$, built with statistical data, we may write the cut functions of the fuzzy Black-Scholes, $\tilde{c}$, on “MIB 30 with strike price 39500, maturity on 11 days”.

**The left cut-function** of $\tilde{c}$ is given by:

$$\tilde{c}^l(y) = \tilde{a}^l(y)N(d_1^l(y)) - 39500e^{-\frac{1}{2}\tilde{\sigma}^2(y)N(d_2^l(y))}$$

where

$$d_1^l(y) = \frac{\log(\frac{\tilde{\delta}^l(y) + \frac{1}{2}(\tilde{\sigma}^l(y))^2}{\tilde{\sigma}^l(y)}\sqrt{\frac{11}{365}}}{\tilde{\delta}^l(y)}$$

and $\tilde{d}_2^l(y) = d_1^l(y) - \tilde{\sigma}^l(y)\sqrt{\frac{11}{365}}$.

**The right cut-function** of $\tilde{c}$ is:

$$\tilde{c}^r(y) = \tilde{a}^r(y)N(d_1^r(y)) - ke^{-t}\tilde{\sigma}^r(y)N(d_2^r(y))$$

where

$$d_1^r(y) = \frac{\log(\frac{\tilde{\delta}^r(y) + \frac{1}{2}(\tilde{\sigma}^r(y))^2}{\tilde{\sigma}^r(y)}\sqrt{\frac{11}{365}}}{\tilde{\delta}^r(y)}$$

and $\tilde{d}_2^r(y) = d_1^r(y) - \tilde{\sigma}^r(y)\sqrt{\frac{11}{365}}$.

Now we have to calculate the basic point of the triangle: the minimum value of the call with belief degree zero, $\tilde{c}^l(0)$, the mean value with belief degree 1, $\tilde{c}^l(1)$ and the max value of the call with belief degree zero $\tilde{c}^r(0)$.

The value of the cut function with the max belief degree, $\tilde{c}^l(1)$, is given by

$$\tilde{c}^l(1) = \tilde{a}^l(1)N(d_1^l(1)) - 39500e^{-\frac{1}{2}\tilde{\sigma}^l(1)N(d_2^l(1))}$$

So we have to calculate $d_1^l(1)$ and $d_2^l(1)$:

$$d_1^l(1) = \frac{\log(\tilde{\delta}^l(1)\sqrt{\frac{11}{365}})}{\tilde{\delta}^l(1)} = \frac{\log(\frac{10000}{39500} + \frac{1}{2}(\tilde{\sigma}^l(1))^2\sqrt{\frac{11}{365}}}{\tilde{\delta}^l(1)} = \frac{\log(1.0126) + \frac{0.02 + (0.25)^2}{2}\sqrt{\frac{11}{365}}}{0.25\sqrt{0.03}} = \frac{0.012578 + 0.05125|0.03}{0.0433} = 0.326$$

with the value of distribution function:

$$N(d_1^l(1)) = N(0.326) = 0.50 + 0.1255 = 0.6255$$

Now we need $d_2^l(1)$:

$$d_2^l(1) = d_1^l(1) - \tilde{\sigma}^l(1)\sqrt{\frac{11}{365}} = 0.326 - 0.0433 = 0.2827$$

with

$$N(d_2^l(1)) = N(0.2827) = 0.50 + 0.11 = 0.61.$$
Substituting these value in \( \hat{c}(1) \) we have:

\[
\hat{c}(1) = 40000 \cdot 0.6255 - 39500e^{-\frac{11}{365} \hat{\delta}^2(1)0.61} = \\
\hat{c}(1) = 25020 - 24095 \cdot e^{-0.03} = \\
\hat{c}(1) = 25020 - 24095 \cdot 0.9994 = \\
= 25020 - 24080.543 = 939.457
\]

So the value of the call with max belief degree is \( \hat{c}(1) = 939.457 \)

For calculating \( \hat{c}(0) \) we need:

\[
\tilde{d}_1(0) = \frac{\log\left(\frac{40000}{39500}\right) + \left[0.014 + \frac{(0.15)^2}{2}\right] \frac{11}{365}}{0.50} = \
\tilde{d}_1(0) = \frac{0.012578 + [0.014 + 0.1125]0.03}{0.50 \cdot 0.17} = \frac{0.012578 + [0.02525]0.03}{0.085} = 0.147 \approx 0.15
\]

and we have

\[
N(\tilde{d}_1(0)) = N(0.15) = 0.50 + 0.0596 = 0.5596
\]

now we need also \( \tilde{d}_2(0) \):

\[
\tilde{d}_2(0) = \tilde{d}_1(0) - \tilde{\sigma}^2(0)\sqrt{\frac{11}{365}} = \\
\tilde{d}_2(0) = 0.15 - 0.50 \cdot 0.17 = \\
= 0.15 - 0.085 = 0.065 \approx 0.06
\]

\[
N(\tilde{d}_2(0)) = N(0.06) = 0.0239 \\
= 0.50 + 0.0239 = 0.5239
\]

Now we can calculate

\[
\hat{c}(0) = \tilde{d}(0) N(\tilde{d}_1(0)) - 39500e^{-\frac{11}{365} \hat{\delta}^2(0)N(\tilde{d}_2(0))} = \\
= 34000 \cdot 0.5596 - 39500e^{-\frac{11}{365} \hat{\delta}^2(0) \cdot 0.5239} = \\
= 19026.4 - 20694e^{-0.03} = \\
= 19026.4 - 20694 \cdot 0.99958 = -1658.959
\]

For calculating \( \hat{c}(0) \) we need \( \tilde{d}_1(0) \) and \( \tilde{d}_2(0) \):

\[
\tilde{d}_1(0) = \frac{\log\left(\frac{\hat{\delta}^2(0)}{39500}\right) + \left[\hat{\delta}^2(0) + \frac{(\hat{\delta}^2(0))^2}{2}\right]0.03}{\hat{\delta}^2(0)\sqrt{0.03}} = \
\tilde{d}_1(0) = \frac{\log\left(\frac{43000}{39500}\right) + [0.025 + \frac{(0.50)^2}{2}]0.03}{0.50\sqrt{0.03}} = \\
= \frac{0.084899 + [0.15]0.03}{0.0866} = \frac{0.084899 + 0.0045}{0.0866} = 1.03
\]
the value of the distribution function is given by

\[ N(1.03) = 0.50 + 0.3485 = 0.8485 \]

\[ \tilde{d}_2(0) = \tilde{d}_1(0) - \tilde{\sigma}(0)\sqrt{0.03} = 1.032 - 0.15 \cdot 0.017 = 1.032 - 0.0255 = 1.00682 \approx 1 \]

the value of the distribution function is given by

\[ N(1) = 0.50 + 0.3413 = 0.8413 \]

so we have the max value:

\[ \tilde{c}_r(0) = 43000 \cdot 0.8485 - 40000 \cdot 0.99925 \cdot 0.8413 = 36485.5 - 33626.76 = 2858.739 \]

So we have a **triangular fuzzy number** Black-Scholes:

\[ \tilde{c}(\omega) = \begin{cases} \frac{\omega - a}{b - a}, & \text{if } \omega \in [a, b] \\ \frac{\omega - b}{c - b}, & \text{if } \omega \in [b, c] \\ 0, & \text{elsewhere.} \end{cases} \]

where \( a = \tilde{c}_l(0) = -1658.959 \), \( b = \tilde{c}_c(1) = 939.457 \), \( c = \tilde{c}_r(0) = 2858.739 \).

The graph of \( \tilde{c} \) looks like a triangle and the support is:

\[ [a, c] = [-1658.959, 2858.739]. \]

The cut functions is given by: \( \forall y \in [0, 1] \)

\[ \tilde{c}^{-l}(y) = a + (b - a)y = -1658.959 + 2598.416 \cdot y \]

\[ \tilde{c}^{-r}(y) = c - (c - b)y = 2858.739 - 1919.282 \cdot y \]

If \( y = 0.90 \) then \( C_y^l = [639.6154, 1131.3852] \);

If \( y = 0.50 \) then \( C_y^u = [-399.751, 1899.098] \)

These evaluation are with the prices of today, November 11th 2006, as we can see on the financial data in internet and on the financial newspapers.

We did these evaluations also in January 1st 2004 and we had a very different **triangular fuzzy number** Black-Scholes:

\[ \tilde{c}(\omega) = \begin{cases} \frac{\omega - a}{b - a}, & \text{if } \omega \in [a, b] \\ \frac{\omega - b}{c - b}, & \text{if } \omega \in [b, c] \\ 0, & \text{elsewhere.} \end{cases} \]

where \( a = -497.91 \), \( b = 197.61 \), \( c = 3384.16 \).

The support of \( \tilde{c} \) was:

\[ [a, c] = [-497.91, 3384.16]. \]

The cut functions of \( \tilde{c} \) were given by: \( \forall y \in [0, 1] \)

\[ \tilde{c}^{-l}(y) = a + (b - a)y = -497 + 695.52 \cdot y \]

\[ \tilde{c}^{-r}(y) = c - (c - b)y = 3384.16 - 3186.55 \cdot y \]
If \( y = 0.90 \) then \( C_y^C = [128.968, 516.265] \);

If \( y = 0.50 \) then \( C_y^C = [149.24, 1790.885] \)

Now, let us compare the supports of our Black-Scholes fuzzy numbers in these two different dates. On January 1st 2004 the support of \( \hat{c} \) was

\[ \Omega_1 = [-497.91, 3384.16]. \]

On November 11th 2006, the support of \( \hat{c} \) was

\[ \Omega_2 = [-1658.959, 2858.739]. \]

The Mib30 is an index of the Italian Stock Exchange so the minimum of these intervals are index of pessimistic performance on the future and the maximum optimistic. We see that (with new data) the pessimistic is become more pessimistic and the optimistic is become less optimistic. The market is becoming with more risks: the risk of loss increase and the forecast of profits decrease.

References